

Use of Quantum Quenches to Probe the Equilibrium Current Patterns of Ultracold Atoms in an Optical Lattice

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Atomic bosons and fermions in an optical lattice can realize a variety of interesting condensed matter states that support equilibrium current patterns in the presence of synthetic magnetic fields or non-abelian gauge fields. As a route to probing such mass currents, we propose a nonequilibrium quantum quench of the Hamiltonian that dynamically converts the current patterns into experimentally measurable real-space density patterns. We illustrate how a specific such “unidirectional” quench of the optical lattice can be used to uncover bulk checkerboard and stripe current orders in lattice Bose superfluids and Fermi gases, as well as chiral edge currents in a quantum Hall state.

Quantum dynamics of particles moving in magnetic fields or non-abelian gauge fields is of great interest in various condensed matter systems such as the cuprate superconductors [1], quantum Hall liquids [2], and topological insulators [3]. Such gauge fields can result in equilibrium charge or spin currents of electrons. For instance, a type-II superconductor in a magnetic field forms an Abrikosov vortex lattice which supports a periodic bulk current pattern [4]. A uniform magnetic field for lattice electrons can lead to topologically nontrivial states with a quantized Hall conductance and chiral edge currents [5]. In a solid, such electronic currents produce their own characteristic magnetic fields, and can thus be probed by using magnetic microscopy [6] or neutron scattering [7].

Recently, experiments have begun exploring effects of “artificial” orbital magnetic fields [8–11] and non-abelian gauge fields [12] on neutral ultracold atomic gases. These experiments can potentially realize a wide variety of fermionic and bosonic states with equilibrium mass currents. This brings us to a crucial question of great interest: *How can we observe the equilibrium mass current patterns for neutral atomic gases?*

In this paper, we propose a dynamical probe of currents, which relies on experimental progress in measuring lattice scale modulations of the atom density. Such density mapping tools include noise correlations [13, 14], Bragg scattering [15], and *in situ* microscopy at the lattice scale [16]. Our key idea is to make a specific quantum quench of the Hamiltonian that violates the steady state *divergence-free* condition on equilibrium currents, thus causing an imbalance between currents entering and leaving different sites. This leads to characteristic density build-up or depletion, as dictated by the continuity equation. Imaging the subsequent density variation across the lattice then yields *real space* information about the initial currents. Our proposal thus links up with another very important thread in ultracold atom research: quench-induced quantum dynamics in many-body systems [17].

Consider a thought experiment in which we start in an equilibrium state with currents and instantaneously turn off the particle hopping amplitude on every bond except one. Then, the instantaneous current \mathcal{J} on that one bond is unaffected, but currents on all other bonds

vanish. A short time δt after this local quench, there will be an accumulated density imbalance $\mathcal{J}\delta t$ between the two sites connected by the unperturbed bond; measuring this yields information on the magnitude and direction of the bond current. A similar such “quasi-local” current probe has been used in a recent experimental study of nonequilibrium dynamics in a 1D Bose gas [18].

In this paper, we study equilibrium current patterns in 2D, focusing on a specific “unidirectional” quench, where we suddenly decrease the particle hopping amplitude in one direction across the entire lattice. This is achieved simply by changing the optical lattice laser intensity along one direction. We show that tracking the subsequent space-time variation of the density for this quench protocol can uncover the underlying current patterns in many interesting cases which have been chosen to show that the method works for both bosons and fermions, and can probe bulk as well as edge currents. Specifically, we illustrate this scheme for (i) a square lattice Bose superfluid in a staggered (checkerboard) magnetic flux pattern, (ii) a square lattice Bose superfluid with a striped magnetic flux pattern, (iii) noninteracting fermions on a square lattice in a staggered magnetic flux, and (iv) the integer quantum Hall state of lattice fermions in a uniform magnetic flux that supports chiral edge currents. Such quenches can also probe the current pattern in the recently studied chiral Bose Mott insulator [19] and spin currents of atomic matter.

Bosons in a checkerboard flux pattern. — Consider interacting bosons on a 2D square lattice, described by the Bose Hubbard model $H_{\text{BH}} = -\sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} b_{\mathbf{r}}^\dagger b_{\mathbf{r}'} + \sum_{\mathbf{r}} V_{\mathbf{r}} b_{\mathbf{r}}^\dagger b_{\mathbf{r}} + \frac{U}{2} \sum_{\mathbf{r}} b_{\mathbf{r}}^\dagger b_{\mathbf{r}}^\dagger b_{\mathbf{r}} b_{\mathbf{r}}$. Here $J_{\mathbf{r},\mathbf{r}'}$ denotes the boson hopping amplitude between sites (\mathbf{r},\mathbf{r}') , U is the on-site repulsion, and $V_{\mathbf{r}}$ is a harmonic trap potential given by $V_{\mathbf{r}} = V_0(x^2 + y^2)$. We take $J_{\mathbf{r},\mathbf{r}'} = J_{\mathbf{r}',\mathbf{r}}^* \neq 0$ only for nearest neighbors, and choose $J_{\mathbf{r},\mathbf{r}+\hat{x}} = J$ and $J_{\mathbf{r},\mathbf{r}+\hat{y}} = J_y \exp(i(-1)^{x+y}\phi/2)$. This yields staggered magnetic fluxes, $\pm\phi$, that pierce the elementary square plaquettes in a checkerboard pattern; a route to realizing such a flux pattern has been proposed previously [21].

For weak interaction, $U \lesssim J, J_y$, we first solve for the equilibrium ground state [22] for $J_y = J$ by numerically minimizing the Gross-Pitaevskii (GP) energy functional

[20] $E_{GP} = -\sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} \Psi_{\mathbf{r}}^* \Psi_{\mathbf{r}'} + \sum_{\mathbf{r}} (\frac{U}{2} |\Psi_{\mathbf{r}}|^4 + V_{\mathbf{r}} |\Psi_{\mathbf{r}}|^2)$, where $\Psi_{\mathbf{r}}$ is the condensate wavefunction at lattice site $\mathbf{r} \equiv (x, y)$. As shown in Fig.1(a) for a system with linear length $L = 22$, $V_0 = 0.05J$, and an average filling factor of $\bar{n} = 1$, this leads to a superfluid ground state with staggered loop currents. The smooth density profile of the ground state reflects the trap potential, but it does *not* reveal the currents induced by the gauge field.

Starting with this ground state, we suddenly decrease J_y from its initial value $J_y^i = J$ to a final value $J_y^f < J$ at time $t = 0$. The subsequent condensate dynamics is described by the time-dependent GP equation [20]

$$i\hbar \frac{\partial \Psi_{\mathbf{r}}(t)}{\partial t} = -\sum_{\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'}^f \Psi_{\mathbf{r}'}(t) + (U |\Psi_{\mathbf{r}}(t)|^2 + V_{\mathbf{r}}) \Psi_{\mathbf{r}}(t). \quad (1)$$

Henceforth, we set $\hbar = 1$. To study the dynamics, we numerically integrate this time-dependent GP equation [22], which allows us to extract the condensate wavefunction $\Psi_{\mathbf{r}}(t)$, and the boson density $|\Psi_{\mathbf{r}}(t)|^2$, at later times. As shown in Fig.1, the condensate at $t > 0$ develops a striking checkerboard *density* pattern that reflects the underlying current order. Information about the direction of circulation on a plaquette is easily discerned from the density pattern established after a short time period; since the quench is along J_y , the initial build up of density is on sites that have currents flowing into them along the strong J -bonds oriented along the x -axis. After a short time has passed, the density buildup reaches a maximum and the flow is reversed, resulting in “plasma oscillations” between the two checkerboard patterns. The frequency of these oscillations scales as $\sim \sqrt{UJ|\Psi_{\mathbf{r}}|^2}$; it varies slowly with position due to the density inhomogeneity in the trap. Such checkerboard oscillations persist for a long time, as inferred from the density difference between the two sublattices $\Delta n_{AB}(t) = n_A(t) - n_B(t)$ shown in Fig.1.

We find that the sublattice density pattern oscillations are also robust against moderate random phase fluctuations imprinted on the initial state [22]. This suggests that thermal phase fluctuations will suppress but not completely destroy these oscillations. Remarkably, the sublattice density oscillation persists out to long times, $tJ \gg 1$, as seen in Fig.1(b). At these times, shown in Fig. 1(e) and (f), we find additional long-wavelength modulations superimposed on the checkerboard density pattern. (i) Spherical density waves emanate periodically outward from the centre, which we attribute to the spatial variation of the “plasma frequency” resulting from the radial variation of the density $|\Psi_{\mathbf{r}}|^2$ in the trap. (ii) The cloud shape shows oscillatory distortions into an ellipse due to the anisotropy of the final tunneling $J_y < J$.

Bosons in a stripe flux pattern. — We next consider the above Bose Hubbard model in the presence of a striped magnetic flux pattern as realized in a recent experiment [9]. We choose $J_{\mathbf{r},\mathbf{r}+\hat{x}} = J$ and $J_{\mathbf{r},\mathbf{r}+\hat{y}} = J_y \exp(i(-1)^y \phi x)$, so that we enclose fluxes $\pm\phi$ through each plaquette that lies along a stripe in the vertical direction, and solve for the equilibrium ground state by minimizing the GP energy functional for $U \lesssim J, J_y$, with

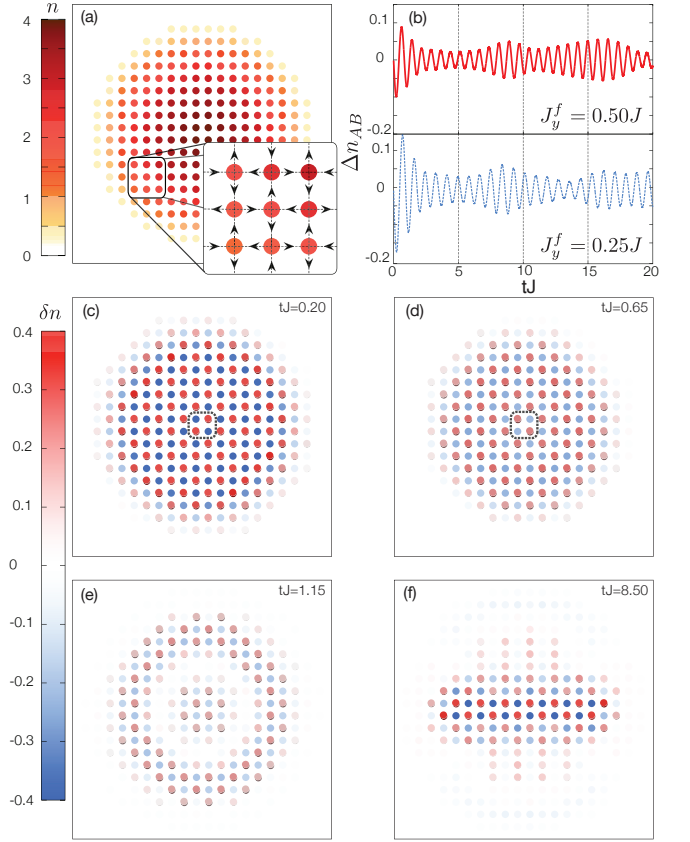


FIG. 1: (Color online) Density pattern of 2D Bose superfluid in a staggered flux gauge field following a quench. (a) Initial density profile and (inset) current pattern of condensate ground state at half-filling. (b) Time dependence of the sublattice density difference $\Delta n(t)$ for a staggered flux of $|\phi| = \pi/2$ flowing quench of $J_y^i = J$ to $J_y^f = 0.25J$ (dashed blue) and $J_y^f = 0.50J$ (solid red). (c - f) Change in local density, δn (relative to original density), at different times following a quench $J_y^i = J \rightarrow J_y^f = 0.50J$ with $U = J$. Circled region in (c), (d) indicates central plaquette vortex.

$L = 22$, $V_0 = 0.05J$, and an average filling $\bar{n} = 1$. We find a superfluid with vertically striped loop currents depicted in Fig.2, which resembles a stripe pattern of ‘long vortices’ that are highly elongated along the y -direction. Again, the smooth equilibrium density pattern reveals *no* information about the underlying current order.

Upon quenching J_y , the superfluid generates a *density* pattern that strikingly reflects the underlying equilibrium striped currents. Each vertically elongated loop forms four quadrants of alternating high and low density, giving rise to an oscillatory quadrupole moment $\sum_{xy}^{\text{loop}} xy \rho(x, y, t)$ about the center of each ‘long vortex’, where the sum is restricted to a single vertically elongated loop composed of square plaquettes. The density pattern oscillates along the top (or bottom) half of the observed density profile as we go along the x -direction. At intermediate times, we again find that the spatial dependence of the oscillation frequency leads to additional long wave-

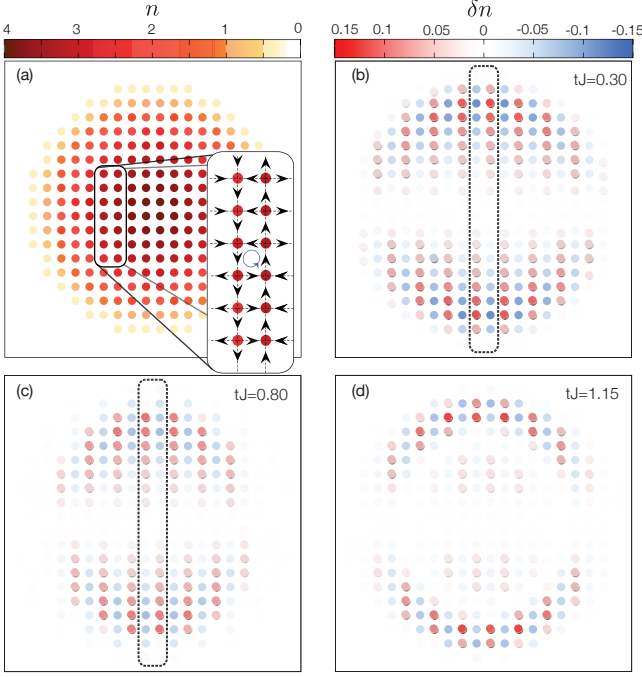


FIG. 2: (Color online) Dynamical density pattern of interacting bosons on a 2D square lattice in a stripe-like magnetic flux pattern following a quench. (a): Initial density profile and (inset) current pattern of condensate ground state at half-filling. (b – e): Change in local density, δn (relative to the original density), at different times following a quench $J_y^i = J \rightarrow J_y^f = 0.50J$ with $U = J$. Circled region indicates the central elongated vortex with a nonzero quadrupole moment.

length stripe-like waves emanating from the trap centre. Quenching J (i.e., along the x -direction) rather than J_y leads to a similar early time density patterns. However the oscillatory dynamics occurs on longer time scales due to mass transport occurring over larger distances $\sim L$.

Spinless fermions in a staggered flux. — Motivated by exploring such quench-induced density dynamics for fermions, we next turn to noninteracting fermions in a staggered flux background [23]. We study the Hamiltonian $H_{\text{sf}} = -\sum_{\mathbf{r}, \mathbf{r}'} J_{\mathbf{r}, \mathbf{r}'} f_{\mathbf{r}}^\dagger f_{\mathbf{r}'}$, where $J_{\mathbf{r}, \mathbf{r}+\hat{x}} = J$ and $J_{\mathbf{r}, \mathbf{r}+\hat{y}} = J_y \exp(i(-1)^{x+y}\phi/2)$, leading to staggered checkerboard fluxes $\pm\phi$ [21]. In momentum space [24], $H_{\text{sf}} = \sum_{\mathbf{k}} \Omega_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger (\cos \theta_{\mathbf{k}} \tau^z + \sin \theta_{\mathbf{k}} \tau^y) \Psi_{\mathbf{k}}$, where $\Psi_{\mathbf{k}}^\dagger = (f_{\mathbf{k}}^\dagger, f_{\mathbf{k}+\mathbf{Q}}^\dagger)$, $\tau^{y,z}$ are Pauli matrices, and the prime on the sum implies that only momenta in the reduced Brillouin zone (BZ) are included. Here, we have defined $\Omega_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \gamma_{\mathbf{k}}^2}$, $\cos \theta_{\mathbf{k}} = \varepsilon_{\mathbf{k}}/\Omega_{\mathbf{k}}$, and $\sin \theta_{\mathbf{k}} = \gamma_{\mathbf{k}}/\Omega_{\mathbf{k}}$, with $\varepsilon_{\mathbf{k}} = -2(J \cos k_x + J_y \cos \frac{\phi}{2} \cos k_y)$ and $\gamma_{\mathbf{k}} = -2J_y \sin \frac{\phi}{2} \cos k_y$. This leads to mode energies $\pm\Omega_{\mathbf{k}}$.

Imagine fermions initially filled into negative energy states $-\Omega_{\mathbf{k}}$ up to a Fermi energy E_F , and then quenching J_y from $J_y^i \rightarrow J_y^f$ at time $t=0$, which instantaneously changes $(\Omega_{\mathbf{k}}, \theta_{\mathbf{k}}) \rightarrow (\tilde{\Omega}_{\mathbf{k}}, \tilde{\theta}_{\mathbf{k}})$. This translationally invariant quench ensures that different momentum pairs $(\mathbf{k}, \mathbf{k}+\mathbf{Q})$ stay decoupled from each other. The ensuing “spin

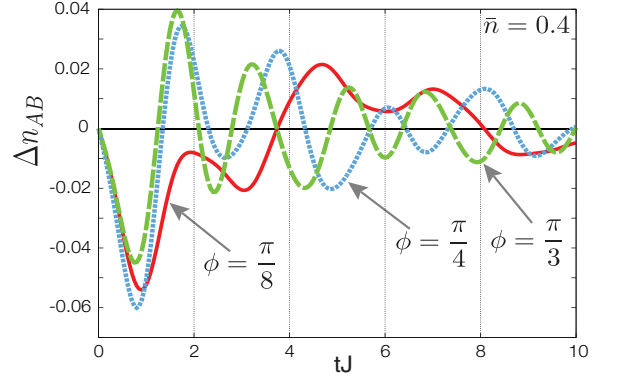


FIG. 3: (Color online) Time dependence of the sublattice density difference $\Delta n_{AB}(t)$, for noninteracting spinless fermions on a 2D square lattice at a filling of $\bar{n} = 0.4$, and various staggered fluxes ϕ , following a quench $J_y^i = J \rightarrow J_y^f = 0.5J$.

precession” type dynamics in momentum space leads to an oscillating density difference between the two sublattices [22] $\Delta n_{AB}(t) = \frac{1}{N} \sum_{\{\mathbf{k}\}_{\text{occ}}} \sin(\theta_{\mathbf{k}} - \tilde{\theta}_{\mathbf{k}}) \sin(2\tilde{\Omega}_{\mathbf{k}}t)$ where the momentum sum runs over only *initially occupied* states in the reduced BZ. A numerical evaluation of the sum allows us to plot the sublattice density oscillations, shown in Fig. 3 for $J_y^i = J$, $J_y^f = 0.5J$, fermion density $\bar{n} = 0.4$ per site, and various staggered flux values.

These oscillations exhibit multiple frequencies due to the large number of occupied fermion modes; the very generation of such a sublattice imbalance is evidence of an initial staggered current pattern. Over the entire range of displayed fluxes, and a wide range of densities $\bar{n} \sim 0.3-0.5$ near half-filling, we find that the dominant oscillation frequency arises from initially occupied states near $\mathbf{k} = (0, \pi)$ due to a van Hove singularity in the density of states [22]. This leads to an estimated, nearly density-independent, dominant oscillation frequency $\tilde{\Omega}^* \approx 2\sqrt{J^2 + (J_y^f)^2 - 2JJ_y^f \cos \frac{\phi}{2}}$, in good agreement with the numerical data in Fig. 3. The weak density dependence of $\Delta n_{AB}(t)$ over a range of fillings indicates that trap induced inhomogeneities will not significantly affect these oscillations.

Lattice quantum Hall state. — Finally, we turn to topologically nontrivial states that support *edge* currents. Recent work has focused on extracting the nontrivial band topology from time-of-flight measurements [25, 28] or spectroscopy of the edge modes [27]. Here we explore density dynamics induced by the unidirectional quench for lattice fermions in a uniform magnetic field. Proposals to obtain such uniform fluxes exist in the literature [18, 28]. For concreteness, consider fermions on a 2D square lattice with a uniform magnetic flux $\phi = 2\pi/3$ per plaquette. The resulting particle-hole symmetric Hofstadter spectrum [29] has three non-overlapping bands, with Chern numbers $+1, -2, +1$, so that ‘band insulators’ with *some* bands being completely filled support a nonzero quantized Hall conductance, and chiral edge

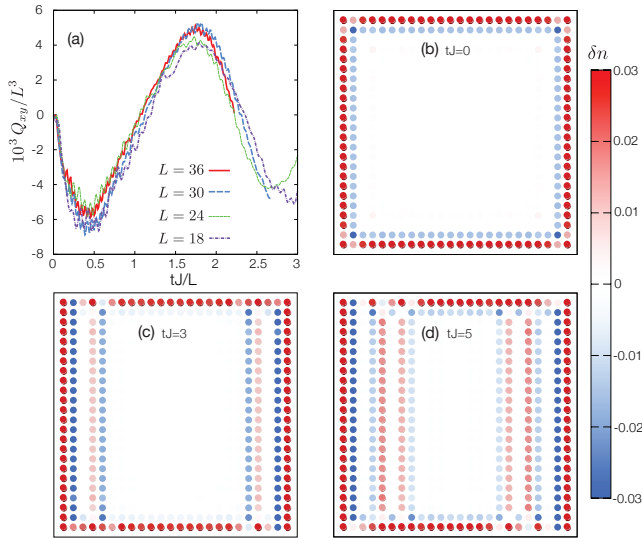


FIG. 4: (Color online) Density dynamics for spinless fermions in the lowest Hofstadter band with flux $\phi = 2\pi/3$ per plaquette on a square lattice following a quench from $J_y^i = J$ to $J_y^f = 0.5J$. (a) The scaled quadrupole moment $Q_{xy}(t)/L^3 = (1/L^3) \sum_{x,y} xy \delta n(x,y,t)$ for various system sizes versus the scaled time tJ/L . (b – d) Stripe-like density modulations (for $L = 24$) moving from the y -edges into the initially incompressible bulk at different indicated times.

currents, yielding lattice quantum Hall (QH) states [5].

We begin by numerically diagonalizing the Hamiltonian $H_{\text{QH}} = -\sum_{\mathbf{r},\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'} f_{\mathbf{r}}^\dagger f_{\mathbf{r}'}$ with $J_{\mathbf{r},\mathbf{r}+\hat{x}} = J$ and $J_{\mathbf{r},\mathbf{r}+\hat{y}} = J_y e^{i\phi x}$, for $\phi = 2\pi/3$, with open boundary conditions on a $L \times L$ system, and fill up the lowest band (and some edge modes) to get a fermion filling $\bar{n} = 1/3$. We find that the ground state bulk density is uniform (see Fig. 4 for $t = 0$) and supports edge currents confined to an ‘‘edge layer’’ of thickness $\sim 2 - 3$ lattice sites. We next track the density dynamics following a quench from $J_y^i = J$ to $J_y^f < J$, which is easy to study once we compute the initial and final spectrum and eigenstates.

Viewing the chiral edge currents as analogous to that arising from a ‘vortex’, we expect the quench to lead to quadrupolar density oscillations and current reversals, similar to what we found for the ‘long vortex’ in the stripe flux superfluid. Inspired by recent work on the superfluid Hall effect of atomic bosons [30], we study the behavior

of the quadrupole moment $Q_{xy}(t) = \sum_{x,y} xy \delta n(x,y,t)$, where the density deviation δn is with respect to $\bar{n} = 1/3$. We find that $Q_{xy}(t)$ indeed displays oscillatory sign reversals and, as seen in Fig. 4(a), the data for various L collapse when plotted as $Q_{xy}(t)/L^3$ versus t/L . The t/L scaling shows that the oscillations occur due to transport across the system length L . A simple scaling argument for an edge current induced oscillation shows that $Q_{xy} \sim L^3$, as we also see numerically. The numerical observations are thus consistent with the quadrupolar oscillations being driven by edge currents. For $\phi = -2\pi/3$, $Q_{xy}(t)$ has the opposite sign. Taken together, these observations provide strong evidence that the initial state is a nontrivial insulator which is *incompressible in the bulk and supports chiral edge currents*.

We observe two additional effects. (i) There are rapid small amplitude oscillations of $Q_{xy}(t)$, over a time scale $\sim J^{-1}$, superimposed on the long time sign reversals. (ii) The quench leads to the dynamic stripe-like density modulations (see Fig. 4) which nucleate near the y -edges and propagate inward into the *bulk* from both directions. We attribute both observations to quench-induced ‘band mixing’. Such mixing occurs across the lattice, but the asymmetry at the edge leads to the stripes originating from the edge. Such mixing with bands of opposite Chern number is also crucial for edge current reversals [22].

Discussion. — We have shown that quantum quenches can yield a probe of underlying current patterns of atoms in an optical lattice by converting them into measurable real-space density orders. A possible concern regarding such sudden quenches is that it may invalidate our tight-binding model since particles can get excited to very high bands of the periodic optical potential. To address this concern, the quench needs to be ‘‘adiabatic’’ on time-scales comparable to the inverse interband gap while being ‘‘sudden’’ on time scales governing intraband dynamics. This seems to be achievable; for instance, time dependent lattice modulation experiments on repulsive atoms in deep lattices can access regimes where they are consistent with theory on the one-band Hubbard model [31]. Looking ahead, it would be extremely useful to explore quench protocols for distinguishing various topological states that may be realized in future experiments.

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